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Convergence and Stability of Modified Random SP-Iteration for A Generalized Asymptotically Quasi-Nonexpansive Mappings

Rashwan Ahmed Rashwan ⋆ *and Hasanen Abuelmagd Hammad*

Abstract

The purpose of this paper is to study the convergence and the almost sure T−stability of the modified SP-type random iterative algorithm in a separable Banach spaces. The Bochner integrability of random fixed points of this kind of random operators, the convergence and the almost sure T−stability for this kind of generalized asymptotically quasi-nonexpansive random mappings are obtained. Our results are stochastic generalizations of the many deterministic results.

Keywords: Almost sure T−stability, separable Banach spaces, random modified SP-iteration, generalized asymptotically quasi-nonexpansive, random fixed point.

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1. Introduction

Random fixed point theory of single-valued and multivalued random operators are the stochastic generalization of the classical fixed point theory for single-valued and multivalued deterministic mappings. The study of random fixed point theory was initiated in 1950 by Prague school of probabilistic. Many authors are impressed by random fixed point theory especially, when Bharucha-Reid [5, 6] presented his papers and his result led to the development of these theorems. Some authors (see, [2, 20, 21, 25]) have shown that under some assumptions, the existence of a deterministic fixed point is equivalent to the existence of a random fixed point. in this case, every deterministic fixed point theorem produces a random fixed point

⋆Corresponding author (Email: rr_rashwan54@yahoo.com)
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theorem. After papers of Bharucha-Reid [5, 6], Špaček [23] and Hanš [9] introduced stochastic extension of the Banach fixed point theorem in a separable metric space. Itoh [10] in 1979, generalized and extended Špaček and Hanš's theorem to a multivalued contraction random operators. Several random fixed point theorems for measurable closed and nonclosed valued multifunctions satisfying general continuity conditions are proved by Papageorgiou [13]. His results improves the results of Engl [8], Itoh [10] and Reich [16].

In 1999, Shahzad and Latif [22] introduced a general random fixed point theorem for continuous random operators. As applications, they stated and proved a number of random fixed points theorems for various classes of 1-set and 1-ball contractive random operators. Rashwan and Hammad [15], proved random fixed point theorems with an application to a random nonlinear integral equation. Chang et al. [7], Beg and Abbas [1] proved some convergence theorems of random Ishikawa scheme and random Mann iterative scheme for strongly pseudo-contractive operators and contraction operators, respectively, in separable reflexive Banach spaces.

Recently, Zhang et al. [26] studied the almost sure $T-$stability and convergence of Ishikawa-type and Mann-type random algorithms for certain $\phi-$weakly contractive type random operators in a separable Banach space. They established the Bochner integrability of random fixed points for this kind of random operators and the almost sure $T-$stability and convergence for these two kinds of random iterative algorithms under suitable conditions. Okeke and Eke [12], extended the results of Zhang et al. [26] by introducing a Noor-type random iterative scheme and studying the same results. These results are the stochastic generalization of the deterministic fixed point theorems of Berinde [3, 4] and Rhoades [17, 18].

In 2011, Phuengrattana and Suantai [14] introduced the following iteration scheme:

Let C be a nonempty subset of a Banach space X

(1) SP-Iteration: Choose $x_1 \in C$ and define

$$\begin{cases} x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n \\ y_n = (1-\beta_n)z_n + \beta_n T z_n \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n \end{cases} \quad n \geq 1, \tag{1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$.

Saluja [19] modified the iteration scheme (1.1) for three generalized asymptotically quasi-nonexpansive self mappings of C as follows:

(2) Modified SP-Iteration: Choose $x_1 \in C$ and define

$$\begin{cases} x_{n+1} = (1-\alpha_n)y_n + \alpha_n T_1^n y_n \\ y_n = (1-\beta_n)z_n + \beta_n T_2^n z_n \\ z_n = (1-\gamma_n)x_n + \gamma_n T_3^n x_n \end{cases} \quad n \geq 1, \tag{1.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$.

Remark 1. If we put $T_1^n = T_2^n = T_3^n = T$ for all $n \geq 1$ in (1.2) then we obtain the SP-iteration (1.1) for generalized asymptotically quasi-nonexpansive self mappings of C.

The aim of this study is to introduce the modified SP-type random iterative scheme and study the convergence and almost sure $T-$stability of our newly random iteration. Also we prove the Bochner integrability of random fixed points for this kind of generalized asymptotically quasi-nonexpansive random mappings in a separable Banach space.

2. Preliminaries

Let (Ω, Σ, μ) is a complete probability measure space where μ be a measure, Σ be a $\sigma-$algebra subset of Ω and C is a nonempty subset of a separable Banach space X.

Definition 2.1. [11] A random variable $x(\omega)$ is Bochner integrable if $\|x(\omega)\| \in L^1(\Omega, \Sigma, \mu)$ implies that

$$\int_\Omega \|x(\omega)\| \, d\mu(\omega) < \infty. \tag{2.1}$$

Proposition 2.2. *[11] A random variable $x(\omega)$ is Bochner integrable iff the sequence of random variables $\{x_n(\omega)\}_{n=1}^{\infty}$ converges strongly to $x(\omega)$ almost surely such that*

$$\lim_{n\to\infty} \int_\Omega \|x_n(\omega) - x(\omega)\| \, d\mu(\omega) = 0. \tag{2.2}$$

Definition 2.3. [26] Assume that (Ω, Σ, μ) is a complete probability measure space and C is a nonempty subset of a separable Banach space X. Let $T : \Omega\times C \to C$ be a random operator and the set $F(T) = \{\zeta(\omega) \in C : T(\omega, \zeta(\omega)) = \zeta(\omega), \omega \in \Omega\}$ denotes the set of random fixed points of T. For any given random variable $x_\circ(\omega) \in C$, define an iterative scheme $\{x_n(\omega)\}_{n=0}^{\infty} \subset C$ by

$$x_{n+1}(\omega) = f(T, x_n(\omega)), \quad n = 0, 1, 2, ... \tag{2.3}$$

where f is a measurable function in the second variable.
Let T has a random fixed point (say $\zeta(\omega)$) which is Bochner integrable with respect to $\{x_n(\omega)\}_{n=0}^{\infty}$. Let $\{y_n(\omega)\}_{n=0}^{\infty} \subset C$ be an arbitrary sequence of a random variable. Assume that

$$\varepsilon_n(\omega) = \|y_{n+1}(\omega) - f(T, y_n(\omega))\| , \tag{2.4}$$

and consider $\|\varepsilon_n(\omega)\| \in L^1(\Omega, \Sigma, \mu)$ where $n = 0, 1, 2, ..$, then the iterative scheme (2.3) is stable with respect to T almost surely ($T-$stable almost surely) iff

$$\lim_{n\to\infty} \int_\Omega \|\varepsilon_n(\omega)\| \, d\mu(\omega) = 0, \tag{2.5}$$

implies that $\zeta(\omega)$ is Bochner integrable with respect to $\{y_n(\omega)\}_{n=0}^{\infty}$.

Definition 2.4. Let C be a nonempty subset of a separable Banach space X and $T : \Omega \times C \to C$ be a random mapping. The random mapping T is said to be:

(i) Nonexpansive if

$$\|T(\omega, x) - T(\omega, y)\| \leq \|x(\omega) - y(\omega)\|,$$

for each $\omega \in \Omega$, where $x(\omega), y(\omega) : \Omega \to C$ are some measurable mappings,

(ii) Quasi-nonexpansive if

$$\|T(\omega, x) - p(\omega)\| \leq \|x(\omega) - p(\omega)\|,$$

for each $\omega \in \Omega$, where $p(\omega) : \Omega \to C$ is a random fixed point of T and $x(\omega) : \Omega \to C$ is any measurable map,

(iii) Asymptotically nonexpansive if there exists a sequence of measurable mappings $\{k_n(\omega)\} : \Omega \to [1, \infty)$ with $\lim_{n\to\infty} k_n(\omega) = 0$ such that

$$\|T^n(\omega, x) - T^n(\omega, y)\| \leq (1 + k_n(\omega) \|x(\omega) - y(\omega)\|,$$

for each $\omega \in \Omega$, where $x(\omega), y(\omega) : \Omega \to C$ are some measurable mappings,

(iv) Asymptotically quasi-nonexpansive if there exists a sequence of measurable mappings $\{k_n(\omega)\} : \Omega \to [1, \infty)$ with $\lim_{n\to\infty} k_n(\omega) = 0$ such that

$$\|T^n(\omega, x) - p(\omega)\| \leq (1 + k_n(\omega) \|x(\omega) - p(\omega)\|,$$

for each $\omega \in \Omega$, where $p(\omega) : \Omega \to C$ is a random fixed point of T and $x(\omega) : \Omega \to C$ is any measurable map,

(v) Generalized asymptotically quasi-nonexpansive if there exists two sequences of measurable mappings $\{r_n(\omega)\}, \{s_n(\omega)\} : \Omega \to [1, \infty)$ with $\lim_{n\to\infty} r_n(\omega) = \lim_{n\to\infty} s_n(\omega) = 0$ such that

$$\|T^n(\omega, x) - p(\omega)\| \leq (1+r_n(\omega) \|x(\omega) - p(\omega)\|+s_n(\omega) \|x(\omega) - T^n(\omega, x(\omega))\|,$$

for each $\omega \in \Omega$, where $p(\omega) : \Omega \to C$ is a random fixed point of T and $x(\omega) : \Omega \to C$ is any measurable map.

We note that a nonexpansive mapping is quasi-nonexpansive but not conversely and a nonexpansive mapping is an asymptotically nonexpansive mapping, an asymptotically nonexpansive mapping is an asymptotically quasi-nonexpansive mapping and an asymptotically quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping.

Inspired and motivated by [19] we introduce a stochastic analogue of the modify iteration scheme (1.2) for three generalized asymptotically quasi-nonexpansive self mappings of C as follows:

(3) Modified Random SP-Iteration:

$$\begin{cases} x_{n+1}(\omega) = (1-\alpha_n)y_n(\omega) + \alpha_n T_1^n(\omega, y_n(\omega)) \\ y_n(\omega) = (1-\beta_n)z_n(\omega) + \beta_n T_2^n(\omega, z_n(\omega)) \\ z_n(\omega) = (1-\gamma_n)x_n(\omega) + \gamma_n T_3^n(\omega, x_n(\omega)) \end{cases} \quad n \geq 1, \tag{2.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$.

The following lemma will be needed in this study.

Lemma 2.5. *[24] Let the sequences $\{a_n\}$ and $\{u_n\}$ of real number satisfy*

$$a_{n+1} \leq (1+u_n)a_n,$$

where $a_n \geq 0$, $u_n \geq 0$, for all $n = 1, 2, \ldots$ and $\sum\limits_{n=1}^{\infty} u_n < \infty$. Then

(i) $\lim\limits_{n\to\infty} a_n$ *exists,*

(ii) if $\lim\limits_{n\to\infty} \inf a_n = 0$, *then* $\lim\limits_{n\to\infty} a_n = 0$.

3. The Convergence Results

Theorem 2.1. *Let $(X, \|.\|)$ be a separable Banach space and $T_1, T_2, T_3 : \Omega \times C \to C$ be three generalized asymptotically quasi-nonexpansive random mappings with two sequences of measurable mappings $\{r_n(\omega)\}, \{s_n(\omega)\} : \Omega \to [1, \infty)$, $\lim_{n\to\infty} r_n(\omega) = \lim_{n\to\infty} s_n(\omega) = 0$ such that $\sum\limits_{n=1}^{\infty} \frac{r_n(\omega)+2s_n(\omega)}{1-s_n(\omega)} < \infty$ and $\bigcap\limits_{i=1}^{3} F(T_i) \neq \varnothing$. Let $\{x_n(\omega)\}$ be a modified SP-random iterative sequence defined by (2.1) where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$ and $x^*(\omega)$ is a common random fixed point of T_1, T_2 and T_3. Then the common random fixed point $x^*(\omega)$ is Bochner integrable provided that $\lim\limits_{n\to\infty} \inf \int\limits_{\Omega} \|x_n(\omega) - x^*(\omega)\| \, d\mu(\omega) = 0$.*

Proof. It suffices to show that

$$\lim_{n\to\infty} \int\limits_{\Omega} \|x_n(\omega) - x^*(\omega)\| \, d\mu(\omega) = 0.$$

For $\omega \in \Omega$, then from (2.1) we get

$$\begin{aligned}
\int_\Omega \|x_{n+1}(\omega) - x^*(\omega)\|\, d\mu(\omega) &\leq (1-\alpha_n)\int_\Omega \|y_n(\omega) - x^*(\omega)\|\, d\mu(\omega) \\
&+ \alpha_n \int_\Omega \|T_1^n(\omega, y_n(\omega)) - x^*(\omega)\|\, d\mu(\omega) \\
&= (1-\alpha_n)\int_\Omega \|y_n(\omega) - x^*(\omega)\|\, d\mu(\omega) \\
&+ \alpha_n \begin{pmatrix} (1+r_n(\omega))\int_\Omega \|y_n(\omega) - x^*(\omega)\|\, d\mu(\omega) \\ +s_n(\omega)\int_\Omega \|y_n(\omega) - T_1^n(\omega, y_n(\omega))\|\, d\mu(\omega) \end{pmatrix} \\
&= (1+\alpha_n r_n(\omega))\int_\Omega \|y_n(\omega) - x^*(\omega)\|\, d\mu(\omega) \\
&+ \alpha_n s_n(\omega)\int_\Omega \|y_n(\omega) - T_1^n(\omega, y_n(\omega))\|\, d\mu(\omega).
\end{aligned}$$

It's obvious that $\alpha_n r_n(\omega) \leq r_n(\omega)$ and $\alpha_n s_n(\omega) \leq s_n(\omega)$, hence

$$\int_\Omega \|x_{n+1}(\omega) - x^*(\omega)\|\, d\mu(\omega) \leq \begin{pmatrix} (1+r_n(\omega))\int_\Omega \|y_n(\omega) - x^*(\omega)\|\, d\mu(\omega) \\ +s_n(\omega)\int_\Omega \|y_n(\omega) - T_1^n(\omega, y_n(\omega))\|\, d\mu(\omega) \end{pmatrix}. \tag{3.1}$$

Now, we have

$$\begin{aligned}
\int_\Omega \|y_n(\omega) - T_1^n(\omega, y_n(\omega))\|\, d\mu(\omega) &\leq \int_\Omega \|y_n(\omega) - x^*(\omega)\|\, d\mu(\omega) \\
&+ \int_\Omega \|T_1^n(\omega, y_n(\omega)) - x^*(\omega)\|\, d\mu(\omega) \\
&\leq \int_\Omega \|y_n(\omega) - x^*(\omega)\|\, d\mu(\omega) \\
&+ (1+r_n(\omega))\int_\Omega \|y_n(\omega) - x^*(\omega)\|\, d\mu(\omega) \\
&+ s_n(\omega)\int_\Omega \|y_n(\omega) - T_1^n(\omega, y_n(\omega))\|\, d\mu(\omega) \\
&= (2+r_n(\omega))\int_\Omega \|y_n(\omega) - x^*(\omega)\|\, d\mu(\omega)
\end{aligned}$$

$$+s_n(\omega)\int_\Omega \|y_n(\omega)-T_1^n(\omega,y_n(\omega))\|\,d\mu(\omega),$$

which implies that

$$\int_\Omega \|y_n(\omega)-T_1^n(\omega,y_n(\omega))\|\,d\mu(\omega) \le \left(\frac{2+r_n(\omega)}{1-s_n(\omega)}\right)\int_\Omega \|y_n(\omega)-x^*(\omega)\|\,d\mu(\omega). \tag{3.2}$$

Using (3.2) and (3.1), it follows that

$$\int_\Omega \|x_{n+1}(\omega)-x^*(\omega)\|\,d\mu(\omega) \le \left(\frac{1+r_n(\omega)+s_n(\omega)}{1-s_n(\omega)}\right)\int_\Omega \|y_n(\omega)-x^*(\omega)\|\,d\mu(\omega). \tag{3.3}$$

Again from (2.1), we can write

$$\begin{aligned}
\int_\Omega \|y_n(\omega)-x^*(\omega)\|\,d\mu(\omega) &\le (1-\beta_n)\int_\Omega \|z_n(\omega)-x^*(\omega)\|\,d\mu(\omega)\\
&+ \beta_n\int_\Omega \|T_2^n(\omega,z_n(\omega))-x^*(\omega)\|\,d\mu(\omega)\\
&\le (1-\beta_n)\int_\Omega \|z_n(\omega)-x^*(\omega)\|\,d\mu(\omega)\\
&+ \beta_n\left(\begin{array}{c}(1+r_n(\omega))\int_\Omega \|z_n(\omega)-x^*(\omega)\|\,d\mu(\omega)\\ +s_n(\omega)\int_\Omega \|z_n(\omega)-T_2^n(\omega,z_n(\omega))\|\,d\mu(\omega)\end{array}\right)\\
&\le (1+r_n)\int_\Omega \|z_n(\omega)-x^*(\omega)\|\,d\mu(\omega)\\
&+ s_n(\omega)\int_\Omega \|z_n(\omega)-T_2^n(\omega,z_n(\omega))\|\,d\mu(\omega).
\end{aligned} \tag{3.4}$$

By the same manner of (3.2), we have

$$\int_\Omega \|z_n(\omega)-T_2^n(\omega,z_n(\omega))\|\,d\mu(\omega) \le \left(\frac{2+r_n(\omega)}{1-s_n(\omega)}\right)\int_\Omega \|z_n(\omega)-x^*(\omega)\|\,d\mu(\omega). \tag{3.5}$$

Applying (3.5) in (3.4), we get

$$\int_\Omega \|y_n(\omega)-x^*(\omega)\|\,d\mu(\omega) \le \left(\frac{1+r_n(\omega)+s_n(\omega)}{1-s_n(\omega)}\right)\int_\Omega \|z_n(\omega)-x^*(\omega)\|\,d\mu(\omega). \tag{3.6}$$

Substituting (3.6) in (3.3), we obtain that

$$\int_{\Omega} \|x_{n+1}(\omega) - x^*(\omega)\| \, d\mu(\omega) \leq \left(\frac{1 + r_n(\omega) + s_n(\omega)}{1 - s_n(\omega)} \right)^2 \int_{\Omega} \|z_n(\omega) - x^*(\omega)\| \, d\mu(\omega). \tag{3.7}$$

Similarly, one can easily write

$$\int_{\Omega} \|z_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \leq \left(\frac{1 + r_n(\omega) + s_n(\omega)}{1 - s_n(\omega)} \right) \int_{\Omega} \|x_n(\omega) - x^*(\omega)\| \, d\mu(\omega). \tag{3.8}$$

From (3.8) and (3.7), it follows that

$$\begin{aligned}
\int_{\Omega} \|x_{n+1}(\omega) - x^*(\omega)\| \, d\mu(\omega) &\leq \left(\frac{1 + r_n(\omega) + s_n(\omega)}{1 - s_n(\omega)} \right)^3 \int_{\Omega} \|x_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&= \left(1 + \frac{r_n(\omega) + 2s_n(\omega)}{1 - s_n(\omega)} \right)^3 \int_{\Omega} \|x_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&= (1 + h_n(\omega))^3 \int_{\Omega} \|x_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\leq (1 + Qh_n(\omega)) \int_{\Omega} \|x_n(\omega) - x^*(\omega)\| \, d\mu(\omega),
\end{aligned}$$

where $h_n(\omega) = \frac{r_n(\omega)+2s_n(\omega)}{1-s_n(\omega)}$ for some $Q > 0$. Putting $u_n = Qh_n(\omega) \geq 0$ in the above inequality and by $\sum\limits_{n=1}^{\infty} \frac{r_n(\omega)+2s_{n(\omega)}}{1-s_n(\omega)} < \infty$, we have $\sum\limits_{n=1}^{\infty} u_n < \infty$. So, since $\lim\limits_{n\to\infty} \inf \int\limits_{\Omega} \|x_n(\omega) - x^*(\omega)\| \, d\mu(\omega) = 0$, all conditions of Lemma 2.1 are satisfied and

$$\int_{\Omega} \|x_n(\omega) - x^*(\omega)\| \, d\mu(\omega) = 0.$$

This complete the proof. □

4. The Stability Results

Theorem 2.1. *Let $(X, \|.\|)$ be a separable Banach space and $T_1, T_2, T_3 : \Omega \times C \to C$ be three generalized asymptotically quasi-nonexpansive random mappings with two sequences of measurable mappings $\{r_n(\omega)\}, \{s_n(\omega)\} : \Omega \to [1, \infty)$, $\lim_{n\to\infty} r_n(\omega) = \lim_{n\to\infty} s_n(\omega) = 0$ such that $\sum\limits_{n=1}^{\infty} \frac{r_n(\omega)+2s_n(\omega)}{1-s_n(\omega)} < \infty$ and $\bigcap\limits_{i=1}^{3} F(T_i) \neq \varnothing$. Let $x^*(\omega)$*

be a random fixed point of T_1, T_2 and T_3 and $\{x_n(\omega)\}$ be a modified SP-random iterative sequence defined by (2.1) converging strongly to $x^*(\omega)$ almost surely, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$. Then $\{x_n(\omega)\}$ is $T-$stable almost surely provided that $\lim\limits_{n\to\infty} \inf \int\limits_{\Omega} \|x_n(\omega) - x^*(\omega)\| \, d\mu(\omega) = 0$.

Proof. Let $\{y_n(\omega)\}_{n=1}^{\infty}$ be any sequence of random variable in C and

$$\|\varepsilon_n\| = \|y_{n+1}(\omega) - (1-\alpha_n)m_n(\omega) - \alpha_n T_1^n(\omega, m_n(\omega))\|, \tag{4.1}$$

where $m_n(\omega) = (1-\beta_n)z_n(\omega) + \beta_n T_2^n(\omega, z_n(\omega))$ and $z_n(\omega) = (1-\gamma_n)y_n(\omega) - \gamma_n T_3^n(\omega, y_n(\omega))$ and $\lim_{n\to\infty} \int\limits_{\Omega} \|\varepsilon_n\| \, d\mu(\omega) = 0$ for every $\omega \in \Omega$. Now we prove that $x^*(\omega)$ is Bochner integrable with respect to the sequence $\{y_n(\omega)\}_{n=1}^{\infty}$. By (4.1)

$$\begin{aligned}
\int\limits_{\Omega} \|y_{n+1}(\omega) - x^*(\omega)\| \, d\mu(\omega) \quad &\leq \quad \int\limits_{\Omega} \|y_{n+1}(\omega) - (1-\alpha_n)m_n(\omega) \\
&\qquad - \alpha_n T_1^n(\omega, m_n(\omega))\| \, d\mu(\omega) \\
&\quad + \quad (1-\alpha_n) \int\limits_{\Omega} \|m_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\quad + \quad \alpha_n \int\limits_{\Omega} \|T_1^n(\omega, m_n(\omega)) - x^*(\omega)\| \, d\mu(\omega) \\
&\leq \quad \int\limits_{\Omega} \|\varepsilon_n\| \, d\mu(\omega) \\
&\quad + \quad (1-\alpha_n) \int\limits_{\Omega} \|m_n(\omega)) - x^*(\omega)\| \, d\mu(\omega) \\
&\quad + \quad \alpha_n \Bigg((1 + r_n(\omega)) \int\limits_{\Omega} \|m_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\quad + \quad s_n(\omega) \int\limits_{\Omega} \|m_n(\omega) - T_1^n(\omega, m_n(\omega))\| \, d\mu(\omega) \Bigg).
\end{aligned}$$

Again since $\alpha_n r_n(\omega) \leq r_n(\omega)$ and $\alpha_n s_n(\omega) \leq s_n(\omega)$, it follows that

$$\int\limits_{\Omega} \|y_{n+1}(\omega) - x^*(\omega)\| \, d\mu(\omega) \leq \left(\begin{array}{c} \int\limits_{\Omega} \|\varepsilon_n\| \, d\mu(\omega) + (1 + r_n(\omega)) \int\limits_{\Omega} \|m_n(\omega)) - x^*(\omega)\| \, d\mu(\omega) \\ + s_n(\omega) \int\limits_{\Omega} \|m_n(\omega) - T_1^n(\omega, m_n(\omega))\| \, d\mu(\omega) \end{array} \right). \tag{4.2}$$

To estimate $\int\limits_{\Omega} \|m_n(\omega) - T_1^n(\omega, m_n(\omega))\| \, d\mu(\omega)$, using (3.2), we get

$$\int_\Omega \|m_n(\omega) - T_1^n(\omega, m_n(\omega))\| \, d\mu(\omega) \leq \left(\frac{2 + r_n(\omega)}{1 - s_n(\omega)} \right) \int_\Omega \|m_n(\omega) - x^*(\omega)\| \, d\mu(\omega). \tag{4.3}$$

Substituting (4.3) in (4.2), we obtain

$$\begin{aligned} \int_\Omega \|y_{n+1}(\omega) - x^*(\omega)\| \, d\mu(\omega) &\leq \int_\Omega \|\varepsilon_n\| \, d\mu(\omega) \\ &+ \left(\frac{1 + r_n(\omega) + s_n(\omega)}{1 - s_n(\omega)} \right) \int_\Omega \|m_n(\omega) - x^*(\omega)\| \, d\mu(\omega). \end{aligned} \tag{4.4}$$

Again to estimate $\int_\Omega \|m_n(\omega) - x^*(\omega)\| \, d\mu(\omega)$, by (3.4), (3.5) and (3.6), we get

$$\int_\Omega \|m_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \leq \left(\frac{1 + r_n(\omega) + s_n(\omega)}{1 - s_n(\omega)} \right) \int_\Omega \|z_n(\omega) - x^*(\omega)\| \, d\mu(\omega). \tag{4.5}$$

Applying (4.5) in (4.4), we get

$$\begin{aligned} \int_\Omega \|y_{n+1}(\omega) - x^*(\omega)\| \, d\mu(\omega) &\leq \int_\Omega \|\varepsilon_n\| \, d\mu(\omega) \\ &+ \left(\frac{1 + r_n(\omega) + s_n(\omega)}{1 - s_n(\omega)} \right)^2 \int_\Omega \|m_n(\omega) - x^*(\omega)\| \, d\mu(\omega). \end{aligned} \tag{4.6}$$

Also to estimate $\int_\Omega \|m_n(\omega) - x^*(\omega)\| \, d\mu(\omega)$, by (3.8), (4.6) we arrive at

$$\begin{aligned} \int_\Omega \|y_{n+1}(\omega) - x^*(\omega)\| \, d\mu(\omega) &\leq \int_\Omega \|\varepsilon_n\| \, d\mu(\omega) \\ &+ \left(\frac{1 + r_n(\omega) + s_n(\omega)}{1 - s_n(\omega)} \right)^3 \int_\Omega \|y_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\ &= \int_\Omega \|\varepsilon_n\| \, d\mu(\omega) \\ &+ \left(1 + \frac{r_n(\omega) + 2s_n(\omega)}{1 - s_n(\omega)} \right)^3 \int_\Omega \|y_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\ &= \int_\Omega \|\varepsilon_n\| \, d\mu(\omega) + (1 + h_n)^3 \int_\Omega \|y_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\ &\leq \int_\Omega \|\varepsilon_n\| \, d\mu(\omega) + (1 + Qh_n) \int_\Omega \|y_n(\omega) - x^*(\omega)\| \, d\mu(\omega), \end{aligned}$$

where $h_n(\omega) = \frac{r_n(\omega)+2s_n(\omega)}{1-s_n(\omega)}$ for some $Q > 0$. Putting $u_n = Qh_n(\omega) \geq 0$ in the above inequality and by $\sum\limits_{n=1}^{\infty} \frac{r_n(\omega)+2s_{n(\omega)}}{1-s_n(\omega)} < \infty$, we have $\sum\limits_{n=1}^{\infty} u_n < \infty$. So, since $\lim_{n\to\infty} \int\limits_\Omega \|\varepsilon_n\| \, d\mu(\omega) = 0$ and $\lim\limits_{n\to\infty} \inf \int\limits_\Omega \|x_n(\omega) - x^*(\omega)\| \, d\mu(\omega) = 0$. Hence all conditions of Lemma 2.1 are satisfied and

$$\lim_{n\to\infty} \int\limits_\Omega \|y_n(\omega) - x^*(\omega)\| \, d\mu(\omega) = 0.$$

Conversely, if $x^*(\omega)$ is Bochner integrable with respect to the sequence $\{y_n(\omega)\}_{n=1}^{\infty}$, then from (4.1), (4.3), (4.4), (4.5) and (4.6), we get

$$\begin{aligned}
\int\limits_\Omega \|\varepsilon_n\| \, d\mu(\omega) &= \int\limits_\Omega \|y_{n+1}(\omega) - (1-\alpha_n)m_n(\omega) - \alpha_n T_1^n(\omega, m_n(\omega))\| \, d\mu(\omega) \\
&\leq \int\limits_\Omega \|y_{n+1}(\omega) - x^*(\omega)\| \, d\mu(\omega) + (1-\alpha_n) \int\limits_\Omega \|m_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&+ \alpha_n \int\limits_\Omega \|T_1^n(\omega, m_n(\omega)) - x^*(\omega)\| \, d\mu(\omega) \\
&\leq \int\limits_\Omega \|y_{n+1}(\omega) - x^*(\omega)\| \, d\mu(\omega) + (1-\alpha_n) \int\limits_\Omega \|y_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&+ \alpha_n(1+k_n(\omega)) \int\limits_\Omega \|m_n(\omega) - x^*(\omega)\| \, d\mu(\omega). \\
&\leq \int\limits_\Omega \|y_{n+1}(\omega) - x^*(\omega)\| \, d\mu(\omega) + (1-\alpha_n) \int\limits_\Omega \|y_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&+ \alpha_n(1+k_n(\omega)) \left(1 + \frac{r_n(\omega)+2s_n(\omega)}{1-s_n(\omega)}\right)^3 \int\limits_\Omega \|y_n(\omega) - x^*(\omega)\| \, d\mu(\omega).
\end{aligned} \tag{4.7}$$

Taking the limit as $n \to \infty$ in (4.7), we get

$$\lim_{n\to\infty} \int\limits_\Omega \|\varepsilon_n\| \, d\mu(\omega) = 0.$$

This shows that the modified random SP-iteration process $\{x_n(\omega)\}_{n=1}^{\infty}$ is $T-$stable almost surely and the conclusion of Theorem 4.2 is proved. □

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R. A. Rashwan
Department of Mathematics,
Faculty of Science,
Assuit University,
Assuit 71516, Egypt
email: rr_rashwan54@yahoo.com

H. A. Hammad
Department of Mathematics,
Faculty of Science,
Sohag University, Sohag
82524, Egypt
email: h_elmagd89@yahoo.com